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Plane Couette Flow of an Incompressible Non-Newtonian Fluid with Temperature Dependent Viscosity

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SUMMARY
It has previously been shown by one of the authors that the variational principle of minimum entropy production with constant fluxes, established by Glansdorff and Prigogine, may be used in presence of convection by defining appropriately the generalized fluxes and forces. In this paper, this principle is applied to the description of the stationary flow of a non-Newtonian viscous incompressible fluid between two parallel plates in relative motion. The fluid is characterized by a viscosity decreasing exponentially with the temperature and by a thermal conductivity independent of the temperature.

The stationary temperature and velocity distributions obtained by the Rayleigh–Ritz variational method are compared with the solutions obtained by direct analytical integration of the conservation equations. Over a large range of the parameters characterizing both the flow and the fluid, a close agreement between the two methods of analysis is obtained even with the simplest trial functions.

1. Introduction

It is well known that for an irreversible process, the entropy production P inside a volume V is a bilinear form for the generalized fluxes J_α and forces X_α

$$P = \int_V J_\alpha X_\alpha dV \,.\,^{**}$$

Prigogine [1] has shown that under some restrictive conditions (linear relations between fluxes and forces, constancy of the phenomenological coefficients, validity of Onsager's reciprocity relations and time-independent boundary conditions), the entropy production corresponding to a purely dissipative process can only decrease with time and reaches its minimum at the stationary state

$$\frac{\partial P}{\partial t} \leqq 0 \,,$$

the equality sign corresponds to the stationary state; in variational notation, Prigogine's principles expresses that

$$\delta P = 0 \quad \text{(stationary state)}\,. \tag{1.1}$$

When the above restrictions are relaxed, Glansdorff and Prigogine [2] have shown that the principle (1.1) remains valid if the fluxes are kept constant:

$$\frac{\partial_X P}{\partial t} = \int_V J_\alpha \frac{\partial X_\alpha}{\partial t} dV \leqq 0 \,,$$

or in variational notation:

$$\delta_X P = \int_V J_\alpha \delta X_\alpha dV = 0 \quad \text{(stationary state)}\,. \tag{1.2}$$

The subscript X indicates that only the forces are submitted to variation.

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** Summation convention on repeated indices is used throughout this work.

Recently, Glansdorff and Prigogine [3], [4] formulated a variational principle describing phenomena including convective processes as well. Under time-independent boundary conditions and with constant fluxes, the principle states that there exists a potential Φ, whose time derivative is always lesser than zero and equal to zero in the stationary state:

$$\frac{\partial_X \Phi}{\partial t} = \int_V J_\alpha \frac{\partial X_\alpha}{\partial t} dV \leqq 0 , \qquad \delta_X \Phi = 0 \quad \text{(stationary state)} . \tag{1.3}$$

Contrary to expressions (1.1) and (1.2), J_α and X_α include now reversible convective contributions. The above criterion has raised many criticisms; they are essentially due to the lack of a clear physical meaning of the functional Φ and to the fact that the construction of Φ is rather artificial and intricate.

Moreover the criterions (1.2) and (1.3) are represented by non total differentials. However, as shown by Glansdorff and Prigogine, these expressions can be linearized in the form of total differentials at least in the neighbourhood of the stationary state; this allows for determining the properties of the stationary states by the methods of the calculus of variation. Several authors (Hays [5], [6], Butler and Rackley [7], Wartique and Nihoul [8]) have applied successfully Glansdorff–Prigogine's principle (1.3) to Couette and Poiseuille flow in the case of Newtonian fluids.

In Glansdorff–Prigogine's meaning, the second principle (1.2) covers exclusively purely dissipative phenomena. However, in absence of chemical reaction and in absence of non-conservative forces, it was shown by Nihoul [9] and Lebon [10], [11] that the criterion (1.2) may be extended to the description of flow processes including reversible contributions. This is done by defining appropriately the generalized fluxes and forces. Following Lebon [10], [11], the fluxes are established from the conservation equations by expressing that their divergence is equal to the rate of local variation of state or kinematical quantities, such as density, velocity, total energy, etc. The generalized forces are determined in the following way: after calculating the expression of the entropy production within a fixed volume containing the fluid, it appears that it exhibits a bilinear form in the fluxes, previously determined, and in some factors defined as being the generalized forces.

The aim of the present work is precisely to apply Lebon's formulation to the problem of the motion of a non-Newtonian viscous incompressible fluid between two parallel planes (Couette flow). The fluid is supposed to be defined by a constitutive power-law, by a constant heat conductivity and an exponential temperature dependent viscosity.

It must be pointed out that our principle is not in contradiction with the results obtained by Gage, Schiffer, Kline and Reynolds [13] on the non-existence of a general variational principle. Their assertion apply to systems characterized by quasi-linear laws between fluxes and forces of the form

$$J_\alpha = L_{\alpha\beta}(\Phi_0 \ldots \Phi_n) X_\beta \tag{1.4}$$

where the forces are defined by

$$X_\alpha = \text{grad}\, \Phi_\alpha , \tag{1.5}$$

the Φ_α being intensive variables such as temperature, pressure, etc. In our formulation, neither (1.4) nor (1.5) are used: our forces are not assumed to be the gradient of intensive variables and there exists no relation between forces and fluxes. Moreover, the results of Gage *et al.* are contradicted by the works of Butler [14] and Keller [15] who were able to formulate variational principles for the class of systems obeying the relations (1.4) and (1.5).

In section 2, Lebon's formulation is briefly recalled and his previous analysis—limited to cartesian coordinates—is extended to the most general vectorial description.

In section 3, it is applied to the Couette flow of a non-Newtonian fluid. After establishing the expression of the functional to be varied, Glansdorff–Prigogine's variational self-consistent method is used to determine the best expressions for the temperature and velocity profiles. These approximate solutions are finally compared with the exact analytical solutions which were obtained by Martin [12] (Section 4).

2. Expression of Glansdorff-Prigogine's Second Criterion in Presence of Convection

In absence of chemical reactions, body forces and heat sources, the conservation equations for mass, momentum and energy may be expressed as:

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \boldsymbol{v}), \tag{2.1}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} = -\operatorname{div}(\rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{\sigma}), \tag{2.2}$$

$$\frac{\partial}{\partial t}\left(\rho \frac{v^2}{2} + \rho u\right) = -\operatorname{div}\left[\boldsymbol{q} + \left(\rho \frac{v^2}{2} + \rho u\right)\boldsymbol{v} - \boldsymbol{\sigma}\cdot\boldsymbol{v}\right]; \tag{2.3}$$

the quantity ρ is the volumic mass; u is the specific internal energy; $\boldsymbol{v}$ is the velocity vector; $\boldsymbol{q}$ the heat conduction vector and $\boldsymbol{\sigma}$ the stress tensor; the above equations have been written in such a form that the local time derivative of a definite quantity be equal to the divergence of an other definite quantity.

Following the definition proposed by Lebon, the generalized fluxes are respectively given by

$$\boldsymbol{J}_0 = \rho\boldsymbol{v}, \tag{2.4}$$

$$\mathbf{J}_1 = \rho\boldsymbol{v}\boldsymbol{v} - \boldsymbol{\sigma}, \tag{2.5}$$

$$\boldsymbol{J}_2 = \boldsymbol{q} + \rho\left(\frac{v^2}{2} + u\right)\boldsymbol{v} - \boldsymbol{\sigma}\cdot\boldsymbol{v}. \tag{2.6}$$

Physically, these fluxes represent respectively the flux of mass, of total momentum and of total energy [16].

In order to establish the expression of the generalized forces corresponding to the fluxes $\boldsymbol{J}_0$, $\mathbf{J}_1$ and $\boldsymbol{J}_2$, let us calculate the entropy production within an arbitrary fixed volume V in the fluid. The corresponding entropy variation is given by

$$\frac{dS}{dt} = \int_V \frac{\partial(\rho s)}{\partial t}\, dV$$

where s is the entropy per unit of mass.

Taking into account the Gibbs relation

$$T ds = du + p d\tau,$$

where τ is the specific volume, T the temperature, p the pressure and performing the calculation as in Lebon's papers [10], [11], one gets

$$\frac{dS}{dt} = \int_V \left[T^{-1}(-\operatorname{div}\boldsymbol{J}_2) - T^{-1}\left(\mu - \frac{v^2}{2}\right)(-\operatorname{div}\boldsymbol{J}_0) - T^{-1}\boldsymbol{v}\cdot(-\operatorname{div}\mathbf{J}_1)\right] dV, \tag{2.7}$$

μ being the chemical potential.

According to the well-known relations

$$\operatorname{div}(m\boldsymbol{a}) = m \operatorname{div}\boldsymbol{a} + \boldsymbol{a}\cdot\operatorname{grad} m,$$

and

$$\operatorname{div}(\mathbf{A}\cdot\boldsymbol{a}) = (\operatorname{div}\mathbf{A})\cdot\boldsymbol{a} + \mathbf{A} : \operatorname{grad}\boldsymbol{a},$$

where a double dot : denotes the inner product of two tensors, expression (2.7) can be written as follows:

$$\frac{dS}{dt} = \int_V \Big\{ \boldsymbol{J}_2 \cdot \text{grad}\, T^{-1} - \text{div}(T^{-1}\boldsymbol{J}_2) + \text{div}\left[T^{-1}\left(\mu - \frac{v^2}{2}\right)\boldsymbol{J}_0\right]$$
$$- \boldsymbol{J}_0 \cdot \text{grad}\left[T^{-1}\left(\mu - \frac{v^2}{2}\right)\right] + \text{div}(T^{-1}\boldsymbol{v}\cdot \mathbf{J}_1) - \mathbf{J}_1 : \text{grad}(T^{-1}\boldsymbol{v})\Big\}\, dV. \quad (2.8)$$

and by application of Gauss' theorem,

$$\frac{dS}{dt} = \int_V \Big\{ \boldsymbol{J}_2 \cdot \text{grad}\, T^{-1} - \boldsymbol{J}_0 \cdot \text{grad}\left[T^{-1}\left(\mu - \frac{v^2}{2}\right)\right] - \mathbf{J}_1 : \text{grad}(T^{-1}\boldsymbol{v})\Big\}\, dV$$
$$+ \int_\Sigma \left[-T^{-1}\boldsymbol{J}_2 + T^{-1}\left(\mu - \frac{v^2}{2}\right)\boldsymbol{J}_0 + T^{-1}\mathbf{J}_1 \cdot \boldsymbol{v}\right] \cdot \boldsymbol{n}\, d\Sigma, \quad (2.9)$$

$\boldsymbol{n}$ being the unit normal pointing outwards to the surface Σ bounding the volume V.

The variation of the total entropy splits into a volume and a surface integral; the latter represents the entropy flux through the surface Σ bounding the volume of the fluid whereas the former is the entropy production P within the volume V. It must be pointed out that P takes the form of a bilinear expression in the generalized fluxes and in some terms which will precisely be defined as the generalized forces; consequently, these are given by

$$\mathbf{X}_0 = -\text{grad}\left[T^{-1}\left(\mu - \frac{v^2}{2}\right)\right], \quad (2.10)$$

$$\mathbf{X}_1 = -\text{grad}(T^{-1}\boldsymbol{v}), \quad (2.11)$$

$$\mathbf{X}_2 = \text{grad}\, T^{-1}. \quad (2.12)$$

It must be observed that the above generalized fluxes and forces are the same as those appearing in the local potential theory of Glansdorff and Prigogine (see [3] and more particularly [17] p. 46). These *generalized* forces and fluxes differ from the usually *thermodynamic* forces and fluxes (16) in that there exist no relations between them.

Furthermore in the case of interest in this work (i.e. a uniform and incompressible fluid) one has

$$\text{div}\, \boldsymbol{J}_0 = 0 \quad (2.13)$$

and, following the reasoning leading from (2.7) to (2.9), it is easy to verify that the only forces which will occur are (2.11) and (2.12).

Let us now show that inequality (1.3) is verified. According to (2.9), the time derivative of the entropy production is given by, when the fluxes are kept fixed,

$$\frac{\partial_X P}{\partial t} = \int_V \Big\{ -\mathbf{J}_1 : \frac{\partial}{\partial t}\text{grad}(T^{-1}\boldsymbol{v}) + \boldsymbol{J}_2 \cdot \frac{\partial}{\partial t}\text{grad}\, T^{-1}\Big\}\, dV. \quad (2.14)$$

Integrating by parts and assuming that the boundary conditions are time-independent, one gets:

$$\frac{\partial_X P}{\partial t} = \int_V \Big\{ \text{div}\, \mathbf{J}_1 \cdot \frac{\partial}{\partial t}(T^{-1}\boldsymbol{v}) - \text{div}\, \boldsymbol{J}_2 \frac{\partial T^{-1}}{\partial t}\Big\}\, dV, \quad (2.15)$$

and taking the conservation laws (2.1) to (2.3) into account:

$$\frac{\partial_X P}{\partial t} = \int \Big\{ -\frac{\partial}{\partial t}(\rho \boldsymbol{v}) \cdot \frac{\partial}{\partial t}(T^{-1}\boldsymbol{v}) + \frac{\partial}{\partial t}\left(\rho \frac{v^2}{2} + \rho u\right) \frac{\partial T^{-1}}{\partial t}\Big\}\, dV. \quad (2.16)$$

Introducing in (2.16), the state equation

$$\frac{\partial u}{\partial t} = C_v \frac{\partial T}{\partial t} \qquad (C_v > 0)$$

where C_v is the specific heat at constant volume, one obtains after some calculations,

$$\frac{\partial_X P}{\partial t} = -\int \left\{ \rho T^{-1} \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial \mathbf{v}}{\partial t} + \rho C_v T^{-2} \left(\frac{\partial T}{\partial t} \right)^2 \right\} dV \leqq 0 \tag{2.17}$$

from which it is clear that $\partial_X P/\partial t$ can only decrease and is equal to zero at the stationary state.

3. Variational Formulation of Couette's Flow Problem

3.1. *Non-Newtonian Fluid Model*

The non-Newtonian fluid is supposed to be described by the following constitutive equation [12]

$$\sigma_{ij} \doteq -p\,\delta_{ij} + 2^{1-2s} C^* \mathrm{e}^{-\alpha(T-T^*)} (I_2)^{-s} d_{ij}\,, \qquad i, j = x, y, z\,. \tag{3.1}$$

This relation is written in Cartesian coordinates and expresses that the stress tensor σ_{ij}, to within the isotropic pressure p, is a function of the rate of strain tensor d_{ij} and the temperature T. The quantity δ_{ij} is the usual Kronecker symbol, T^* is a convenient reference temperature while I_2 is the second invariant of d_{ij}:

$$I_2 = \tfrac{1}{2} d_{ij} d_{ij} \tag{3.2}$$

with

$$d_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{3.3}$$

Finally, C^*, $s(s < 0, 5)$ and $\alpha(\alpha > 0)$ are constants for any fluid obeying the constitutive equation (3.1). The model (3.1) is a special case of the Reiner–Rivlin model in which the coefficient of the rate of strain tensor is assumed to be an exponential function of the temperature. When s is equal to zero, the constitutive equation (3.1) corresponds to a Newtonian fluid whose viscosity η is assumed to have an exponential temperature dependence

$$\eta = C^* \mathrm{e}^{-\alpha(T-T^*)} \tag{3.4}$$

where C^* is the viscosity measured at the reference temperature T^*. It is also assumed that the fluid is incompressible, homogeneous and that the thermal conductivity k is independent of the temperature.

The existence of the Reiner–Rivlin model is questionable. Reiner himself [18] is not sure that fluids obeying the Reiner–Rivlin constitutive equation exist. On the other hand, it is admitted by several authors [12], [19], [20] that the model is useful in describing the behaviour of polymer melts. However, it is not the purpose of this paper to discuss the existence of Reiner–Rivlin's fluids but rather to test the validity of the variational principle $\delta_X P = 0$ by comparing its results with exact ones for certain particular systems.

3.2. *Expression of the Generalized Fluxes and Forces*

Let us now consider the fluid in motion between two infinite parallel plates separated by a distance $2h$, one plate is at rest while the other moves at the velocity U. The cartesian coordinate system is located in a plane midway between the plates, the axes oz and oy being respectively parallel and perpendicular to the direction of the flow.

We are in presence of a one-dimensional problem, the temperature and the velocity fields being respectively of the form $T = T(y)$ and $\mathbf{v} = (0, 0, v_z(y))$ whereas the only non-zero stress component is:

$$\sigma_{zy} = \sigma_{yz} = C^* \mathrm{e}^{-\alpha(T-T^*)} \left(\frac{dv_z}{dy} \right)^{1-2s}. \tag{3.5}$$

Under the above conditions, the expressions of the generalized fluxes and forces are given by:

$$\mathbf{J}_1 = \begin{vmatrix} p & 0 & 0 \\ 0 & p & -\sigma_{yz} \\ 0 & -\sigma_{yz} & \rho v_z^2+p \end{vmatrix}, \tag{3.6}$$

$$J_2 = \left[0, q_y - \sigma_{yz} v_z, \left(\rho \frac{v_z^2}{2} + \rho u + p\right) v_z\right], \tag{3.7}$$

and

$$\mathbf{X}_1 = -\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{d(T^{-1} v_z)}{dy} & 0 \end{vmatrix}, \tag{3.8}$$

$$X_2 = \left(0, \frac{dT^{-1}}{dy}, 0\right), \tag{3.9}$$

while the entropy production is

$$P = \int_V \left\{\sigma_{zy} \frac{d(T^{-1} v_z)}{dy} + (q_y - v_z \sigma_{zy}) \frac{dT^{-1}}{dy}\right\} dV. \tag{3.10}$$

After simplifications, this expression reduces to

$$P = \int \left(\sigma_{zy} T^{-1} \frac{dv_z}{dy} + q_y \frac{dT^{-1}}{dy}\right) dV, \tag{3.11}$$

i.e. the well known expressions for a viscous fluid in which heat conduction is present. In (3.11), the classical thermodynamic forces $T^{-1} dv_z/dy$, dT^{-1}/dy and fluxes σ_{zy}, q_y appear. However this latter expression for P is not suitable for our variational criterion which is only valid when P is exhibiting a bilinear form in the generalized fluxes and forces previously introduced.

3.3. *Variational Formulation*

When the fluxes are kept constant, the first variation of P is equal to zero in the stationary state and given by:

$$\delta_X P = \int_V \left\{\sigma_{zy} \delta \left[\frac{d(T^{-1} v_z)}{dy}\right] + (q_y - \sigma_{zy} v_z) \delta \left[\frac{dT^{-1}}{dy}\right]\right\} dV = 0, \tag{3.12}$$

or after some elementary manipulations,

$$\delta_X P = \int_V \left\{q_y \delta \frac{dT^{-1}}{dy} + \sigma_{zy} \left[\delta \left(T^{-1} \frac{dv_z}{dy}\right) + \frac{dT^{-1}}{dy} \delta v_z\right]\right\} dV = 0. \tag{3.13}$$

The above expression of $\delta_X P$ is not convenient for practical calculations because the variational operator δ does not stand in front of the volume integral. Our task is now to write $\delta_X P$ in the usual form $\delta\Phi$ and to determine the expression of the functional Φ. This will be done by following the same procedure as in Glansdorff and Prigogine's local potential theory.

Call $T_0(y)$ and $v_z^0(y)$ the temperature and velocity distributions corresponding to the stationary state* and replace in (3.13) the quantities q_y, σ_{zy} and σ_{zy} (dT^{-1}/dy) respectively by q_y^0, σ_{zy}^0 and σ_{zy}^0 (dT_0^{-1}/dy). These latter quantities being not subject to variation, expression (3.13) may be written as

$$\delta_X P = \delta \int_V \left\{q_y^0 \frac{dT^{-1}}{dy} + \sigma_{zy}^0 \left[T^{-1} \frac{dv_z}{dy} + \frac{dT_0^{-1}}{dy} v_z\right]\right\} dV = 0. \tag{3.14}$$

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* From now on, all the quantities refering to the stationary state will be labeled with the index "0".

This allows us to define a functional Φ, with $\delta_X P \equiv \delta\Phi$, by

$$\Phi = \int_V \left\{ q_y^0 \frac{dT^{-1}}{dy} + \sigma_{zy}^0 \left[T^{-1} \frac{dv_z}{dy} + \frac{dT_0^{-1}}{dy} v_z \right] \right\} dV \,, \tag{3.15}$$

Φ depends on both variables in the stationary state and out of the stationary state.

Substituting in (3.15) Fourier's law

$$q_y^0 = -k \frac{dT_0}{dy} \tag{3.16}$$

and the constitutive equation

$$\sigma_{zy}^0 = C^* \mathrm{e}^{-\alpha(T_0 - T^*)} \left(\frac{dv_z^0}{dy} \right)^{1-2s} , \tag{3.17}$$

we obtain

$$\Phi = \int_{-h}^{+h} \left\{ -k \frac{dT_0}{dy} \frac{dT^{-1}}{dy} + C^* \mathrm{e}^{-\alpha(T_0 - T^*)} \left(\frac{dv_z^0}{dy} \right)^{1-2s} \left[T^{-1} \frac{dv_z}{dy} + v_z \frac{dT_0^{-1}}{dy} \right] \right\} dy \,. \tag{3.18}$$

Introducing the dimensionless variables

$$W = \frac{v_z}{U} \,, \quad Y = \frac{y}{h} \,, \quad \theta = \frac{\alpha}{Br} T \tag{3.19}$$

with

$$Br = \alpha C^* U^{2-2s} h^{2s}/k \quad \text{(a Brinkman number)} \,,$$

the variational problem to be solved for the determination of the stationary state properties becomes:

$$\delta\Phi \equiv \delta \int_{-1}^{+1} \left\{ \frac{d\theta_0}{dY} \frac{d\theta^{-1}}{dY} - \mathrm{e}^{-Br(\theta_0 - \theta^*)} \left(\frac{dW_0}{dY} \right)^{1-2s} \times \left[\frac{dW}{dY} \theta^{-1} + W \frac{d\theta_0^{-1}}{dY} \right] \right\} dY = 0 \,. \tag{3.20}$$

Note that we do not neglect the operator $W(d/dY)$; therefore and contrary to the works of Hays [5] and Butler–Rackley [7], our analysis is not restricted to slowly moving fluids.

According to (3.15), it is clear that the solution of the variational equations (3.20) leads to the solution of (3.13) at the condition to replace in this latter expression the factors q_y and σ_{zy} by their values in the stationary state.

It is easy to verify that with fixed boundary conditions, the Euler–Lagrange equations corresponding to arbitrary variations of θ^{-1} and W yield the conservation equations of energy and momentum, i.e.

$$\frac{d^2\theta}{dY^2} + \mathrm{e}^{-Br(\theta - \theta^*)} \left(\frac{dW}{dY} \right)^{2-2s} = 0 \,, \tag{3.21}$$

$$\frac{d}{dY} \left[\mathrm{e}^{-Br(\theta - \theta^*)} \left(\frac{dW}{dY} \right)^{1-2s} \right] = 0 \,. \tag{3.22}$$

To obtain these equations, it is necessary to assume that θ_0 and W_0 are not submitted to variation and that at the end of all the operations, the subsidiary conditions $\theta = \theta_0$ and $W = W_0$ are introduced.

3.4. The Self Consistent Method

To solve the variational problem $\delta\Phi=0$, the self-consistent method of Glansdorff and Prigogine is used. This method consists of choosing for θ_0 and θ (respectively for W_0 and W), trial functions satisfying the boundary conditions and having the same dependence on the space variable Y, e.g.

$$\theta_0 = \sum_{n=0}^{N} a_n^0 Y^n, \qquad \theta = \sum_{n=0}^{N} a_n Y^n$$

and

$$W_0 = \sum_{n=0}^{M} b_n^0 Y^n, \qquad W = \sum_{n=0}^{M} b_n Y^n .$$

The coefficients a_n^0 and b_n^0 appearing in the expressions of θ_0 and W_0 are assumed to be known and fixed but those appearing in θ and W are treated as unknowns; these are determinated by the Ritz technique and are given by

$$\frac{\partial\Phi}{\partial a_n} = 0 \quad n=0,1,\ldots,N; \qquad \frac{\partial\Phi}{\partial b_n} = 0 \qquad n=0,1,\ldots,M .$$

After performing these derivations, it is necessary to set

$$a_n^0 = a_n \qquad n=0,1,\ldots,N,$$
$$b_n^0 = b_n \qquad n=0,1,\ldots,M,$$

which is equivalent to applying the subsidiary conditions $\theta_0=\theta$ and $W_0=W$. We are then led to a set of $N+M$ equations for the $N+M$ quantities a_n and b_n.

The analysis of Martin's exact solution shows that the temperature and velocity distributions for different values of the parameters s and Br are rather similar to the corresponding ones observed in the case of a Newtonian fluid. In particular, the temperature distribution θ is an even function of Y while the velocity distribution W is an odd function of Y; further, in the midway plane $Y=0$, W is equal to $\frac{1}{2}$ and θ reaches its maximum. Moreover, let us assume the following boundary conditions:

$$\theta=\theta_1 \ \text{for} \ Y=\pm 1 \tag{3.23}$$

(the plates are at the same uniform temperature), and

$$W=0 \ \text{for} \ Y=-1, \tag{3.24}$$
$$W=1 \ \text{for} \ Y=+1. \tag{3.25}$$

Moreover, for convenience, we choose the reference temperature θ^* equal to the temperature of the plates θ_1.

Under these conditions, appropriate forms for θ and W are

$$\theta = \theta_1+\theta_m(1-Y^2), \tag{3.26}$$
$$\theta_0 = \theta_1+\theta_m^0(1-Y^2) \tag{3.26'}$$

and

$$W = \tfrac{1}{2}(1+Y)+bY(1-Y^2), \tag{3.27}$$
$$W_0 = \tfrac{1}{2}(1+Y)+b_0 Y(1-Y^2). \tag{3.27'}$$

The arbitrary parameters are respectively b and θ_m, which represents the maximum of the temperature profile. According to the Ritz method, the values of the unknowns b and θ_m yielding the best approximating functions W and θ are solutions of the two integral equations:

$$\frac{\partial\Phi}{\partial b} \equiv \int_{-1}^{+1} e^{-Br\theta_0}\left(\frac{dW_0}{dY}\right)^{1-2s}\left[\theta^{-1}\frac{d}{db}\left(\frac{dW}{dY}\right)+\frac{d\theta_0^{-1}}{dY}\frac{dW}{db}\right]dY=0 \tag{3.28}$$

and

$$\frac{\partial \Phi}{\partial \theta_m} \equiv \int_{-1}^{+1} \frac{d\theta_0}{dY} \frac{d}{d\theta_m}\left(\frac{d\theta^{-1}}{dY}\right) dY$$
$$- e^{Br\theta_1} \int_{-1}^{+1} e^{-Br\theta_0} \left(\frac{dW_0}{dY}\right)^{1-2s} \left[\left(\frac{dW}{dY}\right) \frac{d\theta^{-1}}{d\theta_m}\right] dY = 0\,, \tag{3.29}$$

Performing the various derivations, making use of the auxiliary conditions $\theta_m^0 = \theta_m$ and $b^0 = b$, the above expressions become:

$$\int_{-1}^{+1} e^{-Br\theta_m(1-Y^2)} [\tfrac{1}{2} + b(1-3Y^2)]^{1-2s} \left[1 + (1-Y^2)\frac{\theta_m}{\theta_1}\right]^{-1}$$
$$\times \left\{1 - 3Y^2 + 2Y^2(1-Y^2)\frac{\theta_m}{\theta_1}\left[1 + (1-Y^2)\frac{\theta_m}{\theta_1}\right]^{-1}\right\} dY = 0\,, \tag{3.30}$$

$$\int_{-1}^{+1} 4Y^2\theta_m \left[1 + (1-Y^2)\frac{\theta_m}{\theta_1}\right]^{-2} \left\{1 - 2\frac{\theta_m}{\theta_1}(1-Y^2)\left[1 + (1-Y^2)\frac{\theta_m}{\theta_1}\right]^{-1}\right\} dY$$
$$+ \int_{-1}^{+1} e^{-Br\theta_m(1-Y^2)} (Y^2 - 1)[\tfrac{1}{2} + b(1-3Y^2)]^{2-2s} \left[1 + (1-Y^2)\frac{\theta_m}{\theta_1}\right]^{-2} dY = 0\,. \tag{3.31}$$

These equations are of the form

$$F(\theta_m, b) = 0$$
$$G(\theta_m, b) = 0\,.$$

They are solved numerically for various values of the parameters s, Br and θ_1. The quantities θ_m and b are obtained from the Fletcher–Powell minimizing technique. The calculations are performed for one value of θ_1 ($\theta_1 = 1$), for five values of Br ($Br = 10^{-3}$, 10^{-2}, 10^{-1}, 1 and 5) and for five values of s ($s = -0.4$, -0.2, 0, 0.2 and 0.4). The results obtained are presented in table 1. The values of b and θ_m corresponding to $Br = 5$ and $s = 0.4$ cannot be calculated because

TABLE 1

Values of b and θ_m for several values of Br and s

B_r	$s=-0.4$		$s=-0.2$		$s=0$		$s=0.2$		$s=0.4$	
	$10^{+2}b$	θ_m	$10^{+2}b$	θ_m	$10^{+2}b$	θ_m	$10^{+2}b$	θ_m	$10^{+2}b$	θ_m
0.001	0.0007	0.071791	0.0011	0.094727	0.0021	0.124991	0.0046	0.164926	0.0181	0.217633
0.01	0.0066	0.071763	0.0113	0.094680	0.0208	0.124915	0.0458	0.164811	0.1811	0.217593
0.1	0.0662	0.071488	0.1121	0.094218	0.2067	0.124157	0.4534	0.163675	1.7877	0.217134
1	0.6355	0.068895	1.0633	0.089911	1.9287	0.117182	4.1295	0.153102	14.7933	0.207063
5	2.7245	0.059897	4.3720	0.075704	7.4810	0.095166	14.3348	0.119381	—	—

the quantity $\frac{1}{2} + b(1-3Y^2)$ is then negative. It results that for this set of parameters, the trial functions (3.26) and (3.27) have to be replaced by more complicated expressions.

We have also calculated the quantities θ_m and b for another value of θ_1 (namely $\theta_1 = 10$) in the case $Br = 1$. As showed in table 2, the values of θ_m and b are practically insensitive to the values of θ_1.

Substituting the values of θ_m and b in the expressions (3.26) and (3.27), we get the "variational" temperature and velocity profiles; the corresponding numerical results are reported in the next section together with the exact solutions.

TABLE 2

Comparison between the values of b and θ_m calculated respectively for $\theta_1 = 1$ and $\theta_1 = 10$ ($Br = 1$)

s	b		θ_m	
	$\theta_1 = 1$	$\theta_1 = 10$	$\theta_1 = 1$	$\theta_1 = 10$
−0.4	0.006355	0.006356	0.068895	0.068909
−0.2	0.010630	0.010640	0.089911	0.089950
0	0.019287	0.019317	0.117812	0.117297
0.2	0.041295	0.041470	0.153102	0.153511
0.4	0.147393	0.150041	0.207063	0.209609

4. Comparison between the Variational and the Exact Solutions

To determine the accuracy of the variational formulation, an exact solution is required. As established by Martin [12], when the reference temperature θ^* is chosen equal to the plates temperature θ_1, the analytical temperature and velocity distributions satisfying the boundary conditions (3.23), (3.24), (3.25) may be written as

$$\theta = \theta_1 + \frac{2}{B^2} \ln [C^2 \operatorname{sech}^2 (D + C A^{\frac{1}{2}n} BY)] , \tag{4.1}$$

and

$$W = \frac{1}{2} + \frac{A^{\frac{1}{2}n-1} C}{B} \tanh (D + C A^{\frac{1}{2}n} BY) , \tag{4.2}$$

where

$$n = \frac{2-2s}{1-2s} \tag{4.3}$$

and

$$B = \left(Br \, \frac{n-1}{2} \right)^{\frac{1}{2}} ; \tag{4.4}$$

A, D and C are constants of integration given by:

$$D = 0 , \tag{4.5}$$

$$C^2 = \cosh^2 (C A^{\frac{1}{2}n} B) , \tag{4.6}$$

$$C^2 = 1 + \tfrac{1}{4} B^2 A^{2-n} . \tag{4.7}$$

Since the Brinkman number is always positive, it follows from the very definition of B that n must be greater than 1.

Moreover, it can also be shown that the velocity and the temperature profiles are unique in the velocity U of the upper plate, i.e. in the Brinkman number.

The variational and the closed form solutions for $\theta - \theta_1$ are shown in figures 1, 2 and 3 for

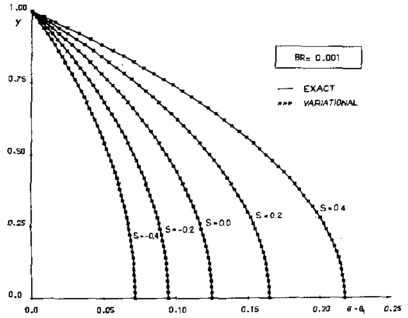

Figure 1. Temperature distribution: Comparison between the variational and the exact solutions.

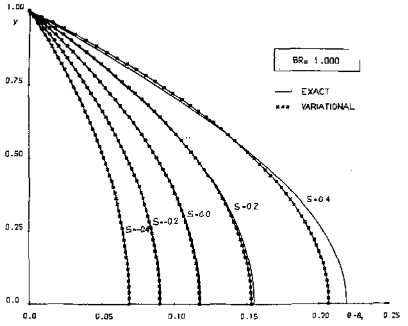

Figure 2. Temperature distribution: Comparison between the variational and the exact solutions.

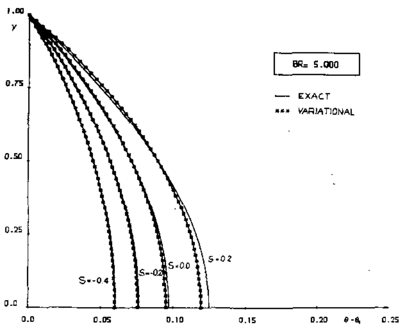

Figure 3. Temperature distribution: Comparison between the variational and the exact solutions.

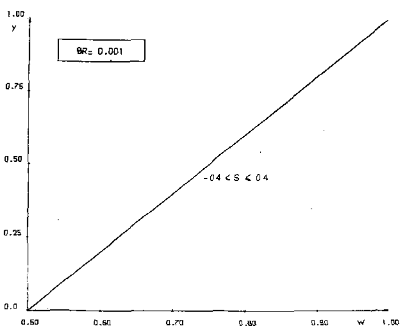

Figure 4. Exact velocity distribution.

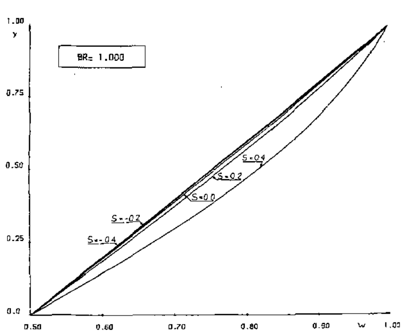

Figure 5. Exact velocity distribution.

three values of Br and for five values of s. In all the calculations θ_1 has been chosen equal to one.

The velocity profiles corresponding to different values of s are very close together; therefore, and in order to maintain clarity in the figures, only the exact solutions are plotted in figures 4, 5 and 6. Comparison between the exact and the variational velocity distributions may be made by examining the graphs of the relative error W(exact) $-$ W(variat.)/W(exact) vs. the dimension-

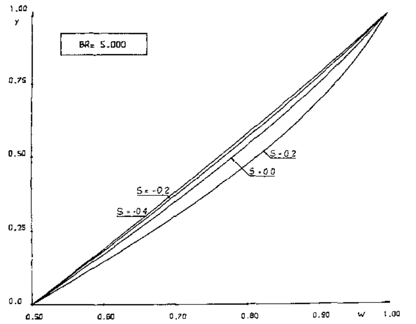

Figure 6. Exact velocity distribution.

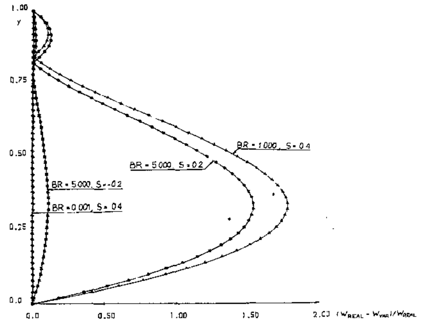

Figure 7. Velocity distribution: Relative error between the variational and the exact solutions.

less coordinate Y presented in figure 7. Although the percent velocity error is not symmetric with respect to Y, only the graphs corresponding to $0 \leqq Y \leqq 1$ are drawn because for negative values of Y and in particular near $Y = -1$, the velocity becomes so small that the notion of relative error is no longer significant.

It is seen that the effect of the Brinkman number on the temperature distribution is weaker than its effect on the velocity distribution. On the contrary, the effect of the parameter s is more important in the case of the temperature profile than in the case of the velocity profile; in particular, for $Br = 10^{-3}$ the velocity distributions corresponding to s varying between -0.4 and 0.4 are not discernable.

It is also apparent from the figures 1, 2, 3 and 7 that the variational solutions agree fairly with the exact ones. For small values of Br, both solutions are practically confused over a wide range of variation of s ($-0.4 \leqq s \leqq 0.4$). Even for the greatest value of Br and s, the variational solutions are always within 10 percent of the exact solution.

It must also be noted that the above results are obtained by choosing very simple forms of the trial functions; in particular, the assumed functions for θ and W do not depend explicitely on the parameters s and Br, i.e. they are written without explicit reference to the non-Newtonian character of the fluid or to the properties of the flow. Therefore, we may conclude that the variational formulation used in this paper yields a simple and powerful technique to solve the non-linear equations describing the flow of non-Newtonian fluids.

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REFERENCES

[1] I. Prigogine, Thèse. *Etude thermodynamique des phénomènes irréversibles*, Liège, Desoer (1947).
[2] P. Glansdorff and I. Prigogine, Sur les propriétés différentielles de la production d'entropie, *Physica*, 20 (1954) 773.

[3] P. Glansdorff and I. Prigogine, On a general evolution criterion in macroscopic physics, *Physica*, 30 (1964) 351.
[4] I. Prigogine and P. Glansdorff, Variational properties and fluctuation theory, *Physica*, 31 (1965) 1242.
[5] D. Hays, A variational formulation applied to Couette and Poiseuille flow, *Bull. Acad. Roy. Sc. Belg.*, 49 (1963) 576.
[6] D. Hays, An extended variational method applied to Poiseuille flow: temperature dependent viscosity, *Int. J. Heat Mass Transfer*, 9 (1966) 165.
[7] H. Butler and R. Rackley, Application of a variational formulation to non equilibrium flows, *Int. J. Heat Mass. Transfer*, 10 (1967) 1255.
[8] J. Wartique and J. Nihoul, Circular Couette motion of an incompressible fluid with temperature dependent viscosity, *Ann. Soc. Sc. de Bruxelles*, 83 (1969) 361.
[9] J. Nihoul, Description hamiltonienne des phénomènes irréversibles stationnaires et éléments d'une théorie de la turbulence, *Journal de Mécanique*, 6 (1967) 171.
[10] G. Lebon, On a variational description of MHD systems including irreversible processes, *J. of Engin. Math.*, 4 (1968) 381.
[11] G. Lebon, Critères d'évolution et processus irréversibles en hydrodynamique, *Bull. Soc. Roy. Sc. de Liège*, 3–4 (1968) 159.
[12] B. Martin, Some analytical solutions for viscometric flows of power-law fluids with heat generation and temperature dependent viscosity, *Int. J. Non Linear Mechanics*, 2 (1967) 285.
[13] D. Gage, M. Shiffer, S. Kline and W. C. Reynolds, The non-existence of a general thermokinetic variational principle, in *Non-equilibrium Thermodynamics, Variational Techniques and Stability*, R. J. Donnelly *et al.*, eds., U. of Chicago Press (1966) 283.
[14] H. Butler, A minimum principle for heat conduction, *Proc. 5th U.S. National Congress of Appl. Mech. A.S.M.E.*, New-York (1966) 739.
[15] J. B. Keller, Extremum principles for Irreversible Processes, *J. of Math. Phys.*, 11 (1970) 2919.
[16] S. R. De Groot and P. Mazur, *Non Equilibrium Thermodynamics*, North-Holland Pub. Co., Amsterdam (1962).
[17] P. Glansdorff, Mechanical flow processes and variational methods based on fluctuation theory, in *Non-equilibrium Thermodynamics, Variational Techniques and Stability*, R. J. Donnelly *et al.*, eds., U. of Chicago Press (1966) 45.
[18] M. Reiner, *Encyclopedia of Physics*, ed. S. Flügge, vol. 6, Springer Verlag, Berlin (1959).
[19] J. Pearson, *Mechanical Principles of Polymer Melt Processing*, Pergamon, New-York (1966).
[20] S. Bostandzhiyan and S. Chernyaeva, Some problems of non isothermal steady flow of a non-Newtonian fluid, *Fluid Dynamics*, 1 (1966) 55.